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Problems in relating various tasks and their sample solutions to Bloom's taxonomy

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ABSTRACT: In this paper we analyze sample solutions of a number of problems and relate them to their level as prescribed by Bloom's taxonomy. We relate these solutions to a number of other frameworks, too. Our key message is that it remains insufficient to analyze written forms of these tasks. We emphasize careful observations of how different students approach a solution before finally assessing the level of tasks used.

We take the arithmetic series as our starting point and point out that the objective of the discussion of the examples here in no way is to indicate an optimal way towards a solution. Instead, our intent is to demonstrate the potential of well selected tasks and the variation that could be observed as students develop their solutions. A large part of this work is devoted to an analysis of the richness of the square triangular number problem.

Keywords: Bloom's taxonomy, Pólya problem solving, square triangular numbers, Diophantine problems, continued fractions.

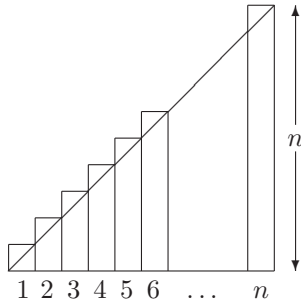


Figure 1: Geometrical derivation of (0.1)

Introduction

We commence by discussing possible solutions of a classical example, more precisely the arithmetic series

$$\begin{aligned}
 s(1) &= 1 \\
 s(2) &= 1 + 2 = 3 \\
 s(3) &= 1 + 2 + 3 = 6 \\
 s(4) &= 1 + 2 + 3 + 4 = 10 \\
 s(5) &= 1 + 2 + 3 + 4 + 5 = 15 \\
 s(6) &= 1 + 2 + 3 + 4 + 5 + 6 = 21 \\
 &\vdots \\
 s(100) &= 1 + 2 + 3 + \cdots + 100 = ?
 \end{aligned}$$

This is trivial computation as long as no methodological question on how to add integer numbers arise. Therefore, adding integer numbers is mathematics as long as methods for performing such procedures with general validity are discussed. After a certain step such difficulties are transformed into routine computation. When such routine computations are needed in order to understand later mathematical structures they need to be fully automatized before they are left (see e.g. Sfard (1992)). We shall refer to this method as inductive experiments as long as its objective is finding governing patterns of the problem in question.

Assuming that adding integer numbers is a procedure classified as routine computation in this context we go on asking whether there are methods for transforming computations of large sums, like $s(100)$ or

$$s(n) = 1 + 2 + 3 + \cdots + n = ?$$

into routine computation? A number of methods obviously exist in this case. First, we notice that these numbers are all triangular, so just by using the involved geometry we get

$$s(n) = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}, \quad (0.1)$$

precisely as in Figure 1.

Another way is to rearrange the sum according to distributive and commutative laws either as

$$\begin{aligned}
 s(100) &= 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100 \\
 &= (1 + 100) + (2 + 99) + (3 + 98) \\
 &= 50 \cdot 101 = 5050
 \end{aligned} \quad (0.2)$$

or

$$\begin{aligned}
 s(101) &= 1 + 2 + 3 + 4 + \cdots + 51 + \cdots + 98 + 99 + 100 + 101 = \\
 &= (1 + 101) + (2 + 100) + (50 + 52) + 51 \\
 &= 50 \cdot 102 + 51 = 100 \cdot 51 + 51 = 101 \cdot 51 = 5151
 \end{aligned} \quad (0.3)$$

and now the recursive relation $s(n+1) = s(n) + n + 1$ is visible, too. It is obvious that (0.2) and (0.3) can be visualized geometrically, too, as we did with (0.1) in Figure 1. We remark that recursive relations are easily derived from explicit relations. The converse direction is more difficult. In this case the recursive relation is linear and therefore easy to solve explicitly by the use of general methods. We show how to solve this problem by the use of this recursive relation, since the solution method contains tools for solving other problems that are related.

The homogenous equation is given by

$$s_H(n+1) - s_H(n) = 0$$

and a test with a trial solution of the type $s_H(n) = \lambda^n$ gives the characteristic equation $\lambda - 1 = 0$. Therefore, the solution of the homogenous equation is a constant $s_H(n) = C \cdot 1^n = C$. The method of undetermined coefficients suggests a particular solution of type $s_P(n) = An + B$, but since this solution agree with one the solutions of the homogenous equation ($A = 0$, $B = C$) we use $s_P(n) = An^2 + Bn$ instead. This technique is often named the replacement rule (Hildebrand (1976)). We substitute this particular solution into the recursive relation and get

$$A(n+1)^2 + B(n+1) - An^2 - Bn = 2An + A + B = n + 1.$$

A consequence of the fundamental theorem of algebra is that the only polynomial equation with infinitely many solutions is the one corresponding to the zero polynomial, so a solution of the above system of equations gives $s(n) = C + \frac{1}{2}n^2 + \frac{1}{2}n$. A suitable initial condition is $s(1) = 1$ which give the value of the constant $C = 0$. Thus, we rediscover (0.1), again.

Taflin (2007) listed seven criteria in order to classify a mathematical problem as rich and these criteria were:

- (i) The problem must introduce important mathematical ideas;
- (ii) The problem must be easy to understand and everybody must have a possibility to work with it;
- (iii) The problem must be experienced as a challenge, require efforts, and time must be assigned for work with it;
- (iv) The problem can be solved in a number of ways by the use of different mathematical ideas and representations;
- (v) Various student's solutions of the problem must initiate discussions that are based on mathematical ideas;
- (vi) The problem must provide links between various fields of mathematics;
- (vii) The problem must initiate formulation of new interesting problems, among students as well as teachers.

I would like to add the possibility that the problem provides links between different students, too, to criterion (vi). Taflin's (2007) Swedish original text allows itself for that possibility. However, from the analysis of the the selected problems in Taflin (2007), it is clear that criterion (vi) is about links between various fields of mathematics and is not about links between different students. On the other hand, such link could be implied by criterion (ii), too.

It is clear that the introductory problem analyzed above has the potential to be a rich problem according to the seven criteria of Taflin (2007). Indeed, Taflin (2007) analyzed a number of problems and two of those: the tower and the stone plates have a mathematical structure that is very similar to our introductory example. These two problems have been the motivation for many Bachelor's and Master's thesis work at teacher's education at Linnaeus University during the period 2010-2016. Our student reports of their observations regarding these problems in their thesis work are one of the motivations to this introductory example. We selected the introductory example here, since it not only contains the structure of these problems, a treatment of its solution facilitates our extensive treatment of the richness of problems associated with square-triangular numbers in Section 3 later on.

It is obvious that this introductory example can be approached in a number of ways, just by computation (ii), using geometry, and using general rules valid for binary operations on numbers (iv). The

example has the potential to raise methodological questions - how are we going to do this (iii). It has the potential to connect several branches of mathematics, apart from arithmetics, geometry, and algebra, the last approach had parallels to methods used analysis and ordinary differential equations (i,vi). What is discovered depends on the guidance the student get from his/her teacher (iii,v,vii).

From our introductory example, we see also that advanced solutions are transformed into routine computations at some stage and we ask at what stage mathematics has been transferred to computations. As long as we are in some process focusing on the methods themselves - how to do this - mathematics is done. On the other hand, the objective of mathematics is transforming matters that at the first glance, gave the impression of being virtually impossible to compute, to routine computations. Some of these routine computations can turn out to be so important for future understanding that they need to be fully automatized (Sfard (1992)). We shall now introduce a number of other examples that can be analyzed at different levels and with a large number of methods intersecting many fields of mathematics.

1 Tools for analyzing the learning outcome

Several tools have been developed for the study of learning events like lessons. If we would observe a lesson including teaching of the example above, we could at least ask the following: (1) In what way does the selected example match the curriculum? (2) Does the selected literature match the curriculum? (3) What critical details are given to the students in the selected literature? (4) What learning objectives does the teacher have when selecting a particular example/approach? (5) What critical details are given by the teacher and how are they related to his/her educational background (Garrison Wilhelm (2014))? (6) What feedback is given by the teacher to the students? (7) How are the learning outcomes/achievements of the students evaluated (Björklund Boistrup (2010) and Hattie and Timperley (2007))? (8) What activities and what outcomes take place at student level (Pólya(1990))? A precise information at this level is central to any evaluation of the activities at any other level.

A number of authors have suggested or used a variety of theoretical frameworks for such analysis, see e. g. Stein, Smith, Henningson, and Silver (2000), Ngware, Ciera, Musyoka (2015), and Bayazit (2013). We want to point out that teacher's decision influence the vast part of all details that can be observed. The teacher does not decide about the curriculum, but selects appropriate literature. The selection of the material in the selected textbook is critical for any future achievement by the students. The teacher is also leading the discussion of the material. All these aspects are heavily influenced by the teachers' mathematical knowledge (Garrison Wilhelm (2014)). One of the most widely used models for identifying cognitive processes is Bloom's taxonomy (cf. Bloom (1984)), but alternatives exist (Gutiérrez, Jaime, Fortuny (1991)). In order to analyze what happens at student level we discuss how it can be developed and applied to mathematics. Bloom's taxonomy has been criticized for being a model for cognitive intentions, but remains far from an accurate description of the cognitive processes that students use when solving achievement test items (Gierl (1997)).

Bloom's taxonomy (cf. Bloom (1984)) in the cognitive domain involved six levels: (1) Knowledge, (2) Comprehension, (3) Application, (4) Analysis, (5) Synthesis, and (6) Evaluation. This does not, however, make sense without examples demonstrating how such levels should be applied to the particular field under study, in this case mathematics. Applied to mathematics the first level (knowledge) could be involved in identifying rules, mastering algorithms, being able to repeat the ordering of binary operations, being able to list properties of real numbers, and redefine concepts that are defined in the used textbook. Knowledge is thus, repeating matters that have previously been encountered in the same form.

Comprehension requires more, it requires translation of concepts and symbols into situations that are different from the setup of the teaching and previous experience. A typical example is forming the algebraic expressions that are needed in problem solving. Two steps are usually involved in the next level, application. First, a symbolic formulation of the problem is needed. Second, manipulations according to algorithms must be done. A mathematical example showing how this could be interpreted is the following problem:

Problem 1.1. By what fraction the area of the rectangle is magnified if the length of its sides is increased by 10%?

The solution of this problem can be written like this: Consider the rectangle to the left in Figure 2.

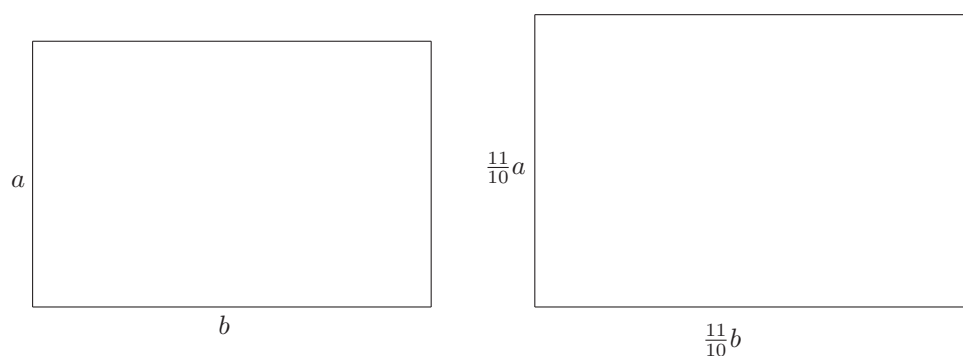


Figure 2: Magnification of a rectangle

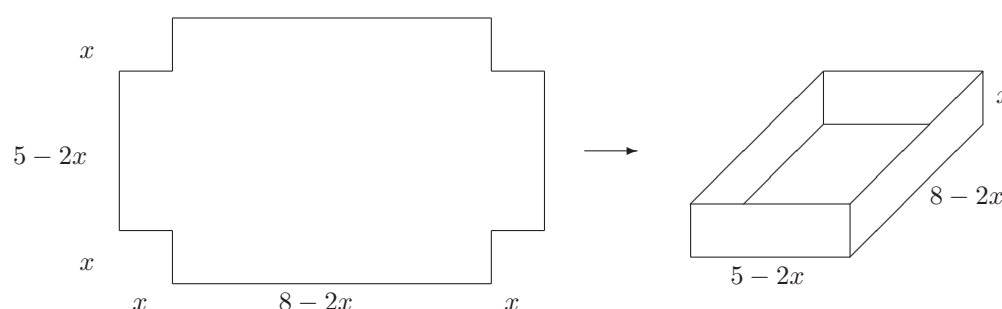


Figure 3: Forming an open box from a rectangle sheet of paper

The area of the first rectangle is ab and the area of the second is

$$\frac{11}{10}a \cdot \frac{11}{10}b = \frac{121}{100}ab.$$

The magnification factor was $\frac{121}{100}$ and we observe that this was invariant of the relation between a and b . After increasing the length of the sides by 10%, the area was increased by 21%.

If area computations have not been trained into routine, this exercise may be graded at the higher levels below, too, since it may involve experiments in an inductive fashion on how to compute areas in general. Finally, we note that the student may start doing inductive experiments before building up an algebraic solution when solving the exercises mentioned here, too. In such cases the mental effort of the involved students can be at still higher level.

Other problems that typically falls in this class are optimization problems usually encountered in high school mathematics (Brijlall and Ndlovu (2013)). A concrete example could be the following problem.

Problem 1.2. A rectangle sheet of paper has sides of length 5 and 8 units, respectively. Construct an open box by cutting its edges according to Figure 3. What corner size maximizes the box's volume?

The solution starts from forming an expression for the volume $V(x)$ and we have

$$V(x) = x(5 - 2x)(8 - 2x) = 40x - 26x^2 + 4x^3, \quad 0 \leq x \leq \frac{5}{2}.$$

In order to find the maximal volume we differentiate and get

$$V'(x) = 40 - 52x + 12x^2 = 3 \cdot 4 \cdot (x - 1)\left(x - \frac{10}{3}\right).$$

It is obvious that $x = 1$ is a maximum for the volume. Moreover, we have $0 < 1 < \frac{5}{2} = \frac{15}{6} < \frac{20}{6} = \frac{10}{3}$. The corner size is therefore given by $x = 1$ and the corresponding maximal volume is

$$V(1) = 40 - 26 + 4 = 18.$$

The grading of the exercise is dependent on the fact that it is encountered in a context where similar problems have been considered or are about to be introduced by the teacher. Meeting this problem without having a reference to what could be an appropriate solution method is at higher level. The construction problem associated with this exercise is at a considerably higher level, too. Which proportions for the side lengths allow for integer/rational solutions of the problem that reduces the amount of (known) algebra involved when training to do such exercises? We shall consider related problems below.

We observe that the grading of the exercise is dependent on the level of the student, too. It is considerably easier and less mathematics involving to complete the previous task when previously experienced patterns are detected. If exercises quite similar to the one selected above has been trained before into routine, it must be degraded to a lower level in the taxonomy. How advanced exercises or activities are and their ability to measure mathematical abilities according to the taxonomy is related to how far it is from previously known material. With respect to certain assumed experiences, this exercise required application, with respect to other; it is a routine computation requiring just knowledge.

2 Higher level tasks

The patterns alluded to above are going to be more evident as we proceed to higher levels. Higher levels are distinguished from the previous ones by, that the number of principles that must be combined in order to solve the problem is larger, the lack of hints to principles or rules that are needed, the discovery of previously unknown relations, and productive thinking rather than reproductive thinking. Productive thinking is promoted only if tasks that can be solved by not previously experienced methods are accessible. We proceed to the next problem for describing the analysis level.

Problem 2.1. Find a positive number t such that $106^2 - t^2$ becomes a prime number.

A solution can be created by first referring to the conjugate rule and we note immediately that any reference to this rule was not given in the exercise. Analysis is by definition the splitting of a larger problem into smaller ones that are connected to the main problem. By the conjugate rule we get

$$106^2 - t^2 = (106 + t)(106 - t)$$

and since t was positive, $t = 105$ since one of the two factors must equal one in order meet the necessary conditions for a prime number. A sufficient condition is that $106 + 105 = 211$ is a prime number. We have that $14 < \sqrt{211} < 15$ and that the only prime numbers below 14 are 2, 3, 5, 7, 11, 13. The number 211 is odd, hence it is not divisible by 2. The sum of the digits $2 + 1 + 1 = 4$, hence it is not divisible by 3. The number 211 does not end with the digit 0 or 5, thus it is not divisible by 5. The remainders of 211 after division by 7, 11, 13 are 1, 2, 3 so 211 is not divisible by any of these numbers and is therefore, a prime number. Typical for the analysis level is the search for a possible solution through a splitting of the problem in smaller problems that can be solved separately. This is also visible here, factorization through conjugate rule, definition of prime numbers, and the division algorithm. Another part that is visible here is that the student is required to analyze and evaluate his solution himself and to be critical against his own thinking. We note here, too, that the solution of this exercise is at knowledge level if the solution already is a routine solution for the student. Again the solution is at the next level if inductive experiments are required before the student is able to formulate the problem algebraically or geometrically.

In this context, we note that fraction arithmetic may appear at analysis-level. Consider the problem:

Problem 2.2. Compute

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{2}{7} + \frac{5}{6}}.$$

Despite that the problem is formulated as a simple computation, the evaluation

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{2}{7} + \frac{5}{6}} = \frac{\frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2}}{\frac{2 \cdot 6}{7 \cdot 6} + \frac{5 \cdot 7}{6 \cdot 7}} = \frac{\frac{3+2}{6}}{\frac{12+35}{42}} = \frac{\frac{5}{6}}{\frac{47}{42}} = \frac{\frac{5}{6} \cdot \frac{42}{47}}{\frac{47}{47} \cdot \frac{42}{47}} = \frac{\frac{5}{6} \cdot \frac{42}{47}}{1} = \frac{5}{6} \cdot \frac{42}{47} = \frac{5 \cdot 6 \cdot 7}{6 \cdot 47} = \frac{35}{47}$$

meets criteria like combination of several principles and that the end result needs to be reduced and checked. It involves splitting of the problem into smaller ones and it involves productive thinking for a student combining all these principles for the first time. There is no need for the example to appear in direct connection to an introduction of the various principles needed for its evaluation and references to these principles do not need to appear in the exercise.

Fraction arithmetic prerequisites have a tremendous impact on the creation of generally valid methods for e. g. equation solving, probabilities, treatment of units in the applied sciences, and the early introduction to limits and derivatives. It is therefore, a paradox to note that problem solving is mentioned at all levels in the Swedish school curricula in mathematics whereas it is never mentioned whether fraction arithmetic at the above level should be automatized into routine or not (Skolverket (2011)). In general, if problem solving is supposed to be trained, it gives no information whatsoever about what prerequisites can be required at the next level, since the content remains unspecified.

Nevertheless, the problems discussed at this level and above meet the different phases of problem solving according to Polya (1970). In general, for problems at analysis level for the student, all four steps from (1) understanding the problem, (2) making a plan, (3) carrying out the plan, to finally (4) looking back on your work are clearly visible.

3 The synthesis level

The synthesis level requires an open search for new relations and production of new results for the student. This is usually hard to test in an exam situation. Many research and thesis work problems fall in this class. In order to describe this level we consider the following problem.

Problem 3.1. Find as many numbers as possible that are square triangular.

For an elementary attempt for starting to understand the problem, we note that we can obtain the following in an inductive/geometric fashion indicated in Figure 4.

Thus, there exists at least two numbers that are triangular and quadratic at the same time, 1 and 36. Adding 0 to that sequence we have 0, 1, 36. It is evident that we are not able to continue producing such numbers in the same manner. Some adjustment of this inductive/geometric method is obviously needed. The formula for triangular numbers (0.1) obtained in Section is now necessary and we need a fully automatized use of this formula in order to proceed. If a number is both triangular and quadratic, then we must have

$$n^2 = \frac{m(m+1)}{2}. \quad (3.1)$$

Since the fundamental theorem of arithmetic guarantees a unique prime number factorization, we have

$$n^2 = n_1^2 n_2^2 = \frac{m(m+1)}{2}.$$

This means that either m or $m+1$ is a square number n_1^2 . If $m = n_1^2$, then $\frac{m+1}{2} = n_2^2$. If $m+1 = n_1^2$, then $\frac{m}{2} = n_2^2$. We collect our results in Table 1

Again, there is a need for revising the method again. We note that, for large numbers n_1 and n_2 , we have

$$n_2 = \sqrt{\frac{n_1^2 \pm 1}{2}} \approx \frac{n_1}{\sqrt{2}}.$$

Moreover, the following questions are still open

1. Is there a finite number of numbers possessing these properties or not?

$n_1^2 n_2^2$	n_2^2	$\frac{n_1^2-1}{2}$	$n_1^2 - 1$	$m = n_1^2$	$n_1^2 + 1$	$\frac{n_1^2+1}{2}$	n_2^2	$n_1^2 n_2^2$
0	0^2	0	0	$0^2 = 0$				
36	2^2	4	8	$1^2 = 1$	2	1	1^2	1
				$2^2 = 4$				
				$3^2 = 9$	10	5		
				$4^2 = 16$				
		12	24	$5^2 = 25$	26	13		
				$6^2 = 36$				
		24	48	$7^2 = 49$	50	25	5^2	1,225
				$8^2 = 64$				
		40	80	$9^2 = 81$	82	41		
				$10^2 = 100$				
		60	120	$11^2 = 121$	122	61		
				$12^2 = 144$				
		84	168	$13^2 = 169$	170	85		
				$14^2 = 196$				
		112	224	$15^2 = 225$	226	113		
				$16^2 = 256$				
41.616	12^2	144	288	$17^2 = 289$	290	145		
				\vdots				
				$29^2 = 841$				
				\vdots				
		840	1,680	$41^2 = 1,681$	1,682	841	29^2	1,413,721
				\vdots				
				$70^2 = 4,900$				
				\vdots				
48,024,900	70^2	4,900	9,800	$99^2 = 9,801$	9,802	4,901		

Table 1: Table of essential results when applying the fundamental theorem of arithmetics to the problem of square triangular numbers

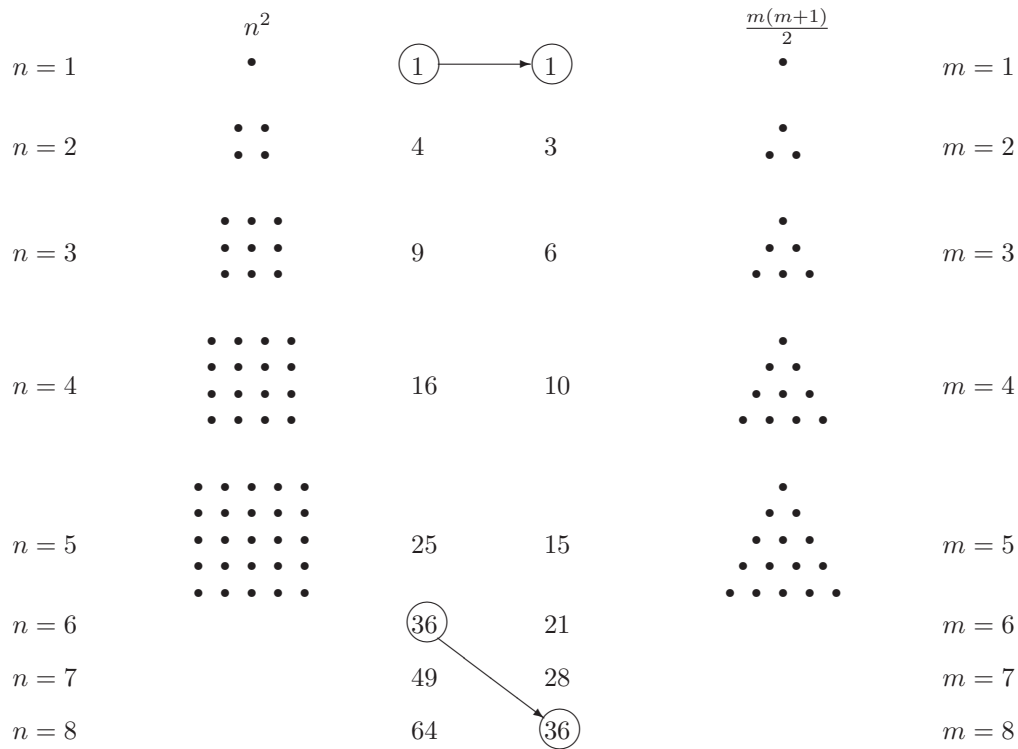


Figure 4: Finding square triangular numbers by geometric plots

i	$n_1(i)$	$n_2(i)$	$n_1^2(i)n_2^2(i)$	$m = n_1^2(i)$	$n = n_1n_2$	$\frac{n_2(i+1)}{n_2(i)}$
1	1	1	1	1	1	2
2	3	2	36	9	6	2.5
3	7	5	1,225	49	35	2.4
4	17	12	41,616	289	204	2.41666..
5	41	29	1,413,721	1,681	1,189	2.41379..
6	99	70	48,024,900	9,801	6,930	\downarrow $1 + \sqrt{2}$

Table 2: Recursive continuation with assistance of analysis and a pocket calculator

2. If there is an infinite amount of such number, is it possible to determine these numbers recursively?
3. The numbers that far discovered in the sequence appear in an alternating manner, odd, divisible by 4, odd, and so on. Are there any missing numbers in this sense?

Such questioning like this is typical when solving problems at higher levels in the taxonomy. Questions should arise naturally as the solution proceeds. In order get everything out of possible elementary solutions and to create some recursive relation we may start by testing whether there are some approximate recursive relations.

Table 2 implies that square triangular numbers behave asymptotically like geometric progression with factor

$$\frac{n_1^2(i+1)n_2^2(i+1)}{n_1^2(i)n_2^2(i)} = (1 + \sqrt{2})^4 = 17 + 12\sqrt{2} \approx 33.97...$$

and that we, with the aid of a pocket calculator may augment the above result table according to

i	$n_1(i)$	$n_2(i)$	$n_1^2(i)n_2^2(i)$	$m = n_1^2(i)$	$n = n_1(i)n_2(i)$
7?	239	169	1, 631, 432, 881	57, 121	40, 391
8?	577	408	55, 420, 693, 056	332, 929	235, 416
9?	1, 393	985	1, 882, 672, 131, 025	1, 940, 449	1, 372, 105

This is how far it seems meaningful to continue generating inductive results. We proceed searching for possible explanations of the above results and observations instead. We start by considering the defining relation (3.1). Multiplying this relation by 8 we get the relation

$$8n^2 = 4m^2 + 4m = (2m + 1)^2 - 1$$

which is equivalent to the Bhaskara-Pell's second degree Diophantine equation usually solved by continued fractions (Selenius (1963)). Keeping the presentation as elementary as possible, we consider

$$x^2 - 2y^2 = 1 \quad (3.2)$$

with $x = 2m + 1$ and $y = 2n$. Starting by $m = n = 1$, we get $x_1 = 3$, $y_1 = 2$, solving (3.2) and a continued fraction expansion of $\sqrt{2}$ would have produced the first solution in case it was unknown. In fact, we have

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\frac{1}{\sqrt{2}-1}} = 1 + \frac{1}{2 + \frac{3-2\sqrt{2}}{\sqrt{2}-1}} = 1 + \frac{1}{2 + \frac{1}{\frac{1}{\sqrt{2}-1}}} = 1 + \frac{1}{2 + \frac{1}{2+\dots}}$$

so the continued fraction expansion of $\sqrt{2}$ is $[1, 2, 2, 2, \dots]$ and repeat itself as it should do for quadratic irrationals. The first two convergents can now be computed

$$1 + \frac{1}{2} = \frac{3}{2} \text{ and } 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{2}{5} = \frac{7}{5}$$

and identified. Now, if we have one solution to this equation (x_i, y_i) then the conjugate rule gives

$$1 = (x_i + y_i\sqrt{2})(x_i - y_i\sqrt{2})$$

and squaring this relation gives

$$\begin{aligned} 1 &= 1^2 = (x_i + y_i\sqrt{2})^2(x_i - y_i\sqrt{2})^2 \\ &= (x_i^2 + 2y_i^2 + 2\sqrt{2}x_iy_i)(x_i^2 + 2y_i^2 - 2\sqrt{2}x_iy_i) \\ &= (x_i^2 + 2y_i^2)^2 - 2(2x_iy_i)^2. \end{aligned}$$

Therefore, the equation is left invariant with respect to this procedure and another solution is given by $(x_i^2 + 2y_i^2, 2x_iy_i)$. There is certainly an infinite amount of square triangular numbers and we ask whether this solution is (x_{i+1}, y_{i+1}) , the next number in the sequence of square triangular numbers? It turns out that this procedure will jump over some of the square triangular numbers, but that a slight modification will give an explicit formula. That is starting from $x_1 = 3$ and $y_1 = 2$, we get by the binomial formula

$$\begin{aligned} 1 &= 1^k = (3 + 2\sqrt{2})^k(3 - 2\sqrt{2})^k \\ &= \left(\sum_{i=0}^k \binom{k}{i} 3^{k-i} (2\sqrt{2})^i \right) \left(\sum_{i=0}^k \binom{k}{i} 3^{k-i} (-)^i (2\sqrt{2})^i \right) \\ &= \left(\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} 3^{k-2i} 2^{3i} \right)^2 - 2 \left(\sum_{i=1}^{\lfloor (k-1)/2 \rfloor + 1} \binom{k}{2i-1} 3^{k-2i+1} 2^{3i-2} \right)^2 \end{aligned}$$

where the last terms are arranged by the conjugate rule. The original equation is left invariant here, too. All the solutions mentioned above can therefore be derived from

$$(x_k, y_k) = \left(\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} 3^{k-2i} 2^{3i}, \sum_{i=1}^{\lfloor (k-1)/2 \rfloor + 1} \binom{k}{2i-1} 3^{k-2i+1} 2^{3i-2} \right)$$

by the translation formulas $m = (x_k - 1)/2$ and $n = y_k/2$. It is then possible to use either the values for m or n to construct square triangular number by use of (3.1).

We have not yet proved that there are no other square triangular numbers than those listed here, but this can be proved. In fact, there were an infinite number of square triangular numbers and it was possible to provide an explicit formula for them. We have now transformed what seemed impossible at the first glance to computation. The task given was at synthesis level, it required collecting of results in an inductive way in order to see the relations first. After solving the task it is at knowledge level, ie it requires repetition of matters that we have previously met in the same form. Mathematics is transforming material at higher levels in the taxonomy to lower levels in the taxonomy. If we would have been able to solve the last problem directly by analytical methods, then it would not have been classified as a synthesis problem. We remark that there are still open research questions related to square triangular numbers, see Keskin and Karaath (2011) and references therein. In this context, it was important to carry out calculations that were not part of a complete solution at the end.

4 The evaluation level

At the evaluation level, it is understood that the solution or non-solution of the problem has consequences for our understanding of the reality in nearby fields and the choice of methods to approach the problems in general. Invention of integral calculus from attempting to understand how to evaluate sums like

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{n^3}$$

is, perhaps the most famous example. Integral calculus transforms not only the evaluation of limits of such sums into the triviality,

$$\begin{aligned} \frac{1}{3} - \frac{1}{2n} \left(1 - \frac{1}{3n}\right) &= \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 < \int_0^1 x^2 dx = \\ &= \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} < \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \end{aligned}$$

but changes our use such expressions, too. We can create another example by generalizing the square triangular number problem. We ask first whether it is possible to find numbers that are simultaneously pyramidal and cubic, ie

$$n^3 = \frac{m(m+1)(2m+1)}{6} = \sum_{k=1}^m k^2 = 1 + 2^2 + 3^2 + \dots + m^2. \quad (4.1)$$

The second equality can be established by mathematical induction as usual. This third degree Diophantine equation has at least one solution, $m = n = 1$, and one may verify that a possible next solution has at least the property that $n > 20$ implying that a possible next cubic pyramidal number must be larger than 8000. This equation has obviously at least one solution, but we remark that finding a systematic method for determining whether general Diophantine equations has solutions or not was Hilbert's (1902) tenth problem. David Hilbert was the last mathematician who was able to keep together a seminar covering all contemporary fields of mathematics and all the problems in his list has had a remarkable impact on the development of mathematics during the last century. Some of them still remain unsolved. The difficulties in proving Fermat's last theorem, cf. Wiles (1995) and Taylor and Wiles (1995), are related to Hilbert's tenth problem.

Hilbert's tenth problem has been proved unsolvable, see e. g. Matiyasevič (1970,1971), the review by Davis (1973), and the recent paper by Matiyasevič (2011). Thus, there exists no finite step method for determining whether a general Diophantine equation has solutions or not. This means that for any proposed method, either a Diophantine equation exists such that the method delivers a false answer or never ends. This does not exclude the possibility that one could determine whether a given special case has solutions or not, but a finite step method to determine whether a general Diophantine equation has solutions or does not exist. Such results have the potential to change our perspectives on what is possible to do and what is not. They have a decisive impact on what methods we choose to include in the curricula for future generations.

5 Conclusion and Remark

We have considered a number of classical problems for describing the different levels in Bloom's taxonomy in mathematics and one of the objectives in this paper was to relate these levels to students' previous knowledge and experience. Our main division of the problems is the one in lower- and higher levels, respectively. We defined higher level problems in general to require a large number of principles that must be combined in order to solve the problem. Furthermore, productive thinking and discovery of previously unknown relations is required.

In many of the problems we analyzed, it is visible that the environment shaped around the student in particular by his/her teacher has a huge impact on the outcome and we note that some of the criteria that Taflin (2007) used for defining rich problems were, at least in part, dependent on action of the teacher in charge of the lecture in question.

The solution of the higher level problems makes it clear that the solution itself has the ability to change the level of the problem. This includes the possibility that a mathematical problem about structure and causal relations might be transformed to a triviality, a computation. Therefore, the problems are at different levels depending on the background of the person considering them. As problems at higher cognitive levels are posed, many alternative ways of solving them are usually posed and new questions are formulated. The problem becomes rich (Taflin (2007)). The question of whether there is a complete solution to the problem is now and then posed.

The level of the problem is dependent on the student's prerequisites and previous experiences and as an extreme example we considered fraction arithmetic. The number of rules and principles that must be combined in order to create an efficient solution of such a problem can be large and simultaneously creating solutions to such problems need to be fully automatized in order to make areas like general methods for equation solving, derivatives, and efficient handling of units in applied sciences accessible.

It is important to have exercises that requires searching for appropriate methods in the curriculum. On the other hand, there are a number of problems that have been considered for decades or by generations of mathematicians before they were solved. It is not reasonable that the students discover all methods themselves. Methods and method development form always an important part of contemporary mathematics must always be included in the curriculum. It is most important to give future students at all levels the tools they need in order to use, understand, and even make mathematics that do not yet exist (cf. Cuoco, Goldenberg, and Mark (1996)).

Problems at synthesis level and above are in most cases not suitable in for testing students in exam situations when the assigned time for work with the question is limited (Taflin (2007)). But problems at synthesis level should be used for defining thesis and related work at various levels.

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